Examen Méca 1 INP
08/03/11

0,5 1. (a) $\omega = 2\pi f = \boxed{\dfrac{2\pi}{T}}$

0,5 $\dfrac{r}{R} = \sin\alpha \Rightarrow \boxed{r = R\sin\alpha}$

0,5 (b) $\vec{OM} = r\,\vec{u_r}$

1 $\vec{v}(M/F) = \left.\dfrac{d\vec{OM}}{dt}\right|_F = \underbrace{\dot{r}}_{0}\,\vec{u_r} + r\dfrac{d\vec{u_r}}{dt} = r\omega\,\vec{u_\varphi}\ \boxed{= \omega R\sin\alpha\,\vec{u_\varphi}}$

1 $\vec{a}(M/F) = \dfrac{d\vec{v}(M/F)}{dt} = \underbrace{\dot{r}}_{0}\,\omega\,\vec{u_\varphi} + r\omega\dfrac{d\vec{u_\varphi}}{dt} = -r\omega^2\vec{u_r}$

$\boxed{= -\omega^2 R\sin\alpha\,\vec{u_r}}$

0,5 (c) $\sum\vec{F} = \vec{P} + \vec{T} = -m\vec{g} + \vec{T}$.

$\vec{P} = -mg\,\vec{u_z}$

$\vec{T} = -T\sin\alpha\,\vec{u_r} + T\cos\alpha\,\vec{u_z}$

0,5 $\boxed{= T\begin{vmatrix} -\sin\alpha \\ 0 \\ \cos\alpha \end{vmatrix}}$

d) $m\vec{a}(M/F) = \sum\vec{F}$ 0,5

$\Leftrightarrow m\begin{vmatrix} -\omega^2 R\sin\alpha \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} -T\sin\alpha \\ 0 \\ -mg + T\cos\alpha \end{vmatrix}$

$\Leftrightarrow \begin{vmatrix} m\omega^2 R = T \\ mg = T\cos\alpha \end{vmatrix}$ 1

(e) $m\omega^2 R = T$

1 et $T\cos\alpha = mg \Rightarrow \cos\alpha = \dfrac{mg}{T} = \dfrac{mg}{m\omega^2 R} = \boxed{\dfrac{g}{\omega^2 R}}$

(f) 1 $E_K = \dfrac{1}{2} m v^2(M/F) = \dfrac{1}{2} m(\omega R\sin\alpha)^2 = \dfrac{1}{2} m\omega^2 R^2\left[1 - \cos^2\alpha\right]$

1 $= \dfrac{1}{2} m\omega^2 R^2\left[1 - \left(\dfrac{g}{\omega^2 R}\right)^2\right] \Rightarrow \boxed{E_0 = \dfrac{1}{2} m\omega^2 R^2}$

(g) Th. de l'Energie cinétique :

1 $W = \dfrac{1}{2} mR^2 \left\{ \omega_2^2\left[1 - \left(\dfrac{g}{R\omega_2^2}\right)^2\right] - \omega_1^2\left[1 - \left(\dfrac{g}{R\omega_1^2}\right)^2\right]\right.$

2

1 1

2 (a) $m\vec{a}(M/R) = \sum \vec{F} = \vec{P} + \vec{T} + \vec{F_{ie}} + \vec{F_{ic}}$ avec $\vec{F_{ie}} = -m\vec{a_e}(M, R/F)$

et $\vec{F_{ic}} = -m\vec{a_c}(M, R/F)$

ou $m\vec{a}(M/F) = \sum \vec{F} = \vec{P} + \vec{T}$ et $\vec{a}(M/F) = \vec{a}(M/R) + \vec{a_e}(M, R/F) + \vec{a_c}(M, R/F)$

1 1

(b) $\vec{v}(M/R) = \left.\dfrac{d\vec{OM}}{dt}\right|_R = \dfrac{d}{dt}(r\vec{e_1}) \boxed{= \vec{0}}$ car $\dot{r} = 0$ et $\dfrac{d\vec{e_1}}{dt} = \vec{0}$ dans R.

0,5

$\vec{a}(M/R) = \dfrac{d\vec{v}(M/R)}{dt} = \boxed{\vec{0}}$ 0,5

(c) $\vec{a_c} = 2\vec{\Omega}_{R/F} \wedge \vec{v}_{M/R} \boxed{= \vec{0}}$. 1

$\Rightarrow \vec{a_e}(M, R/F) = \vec{a}(M/F)$.

O fixe — O en vitesse uniforme — $\omega\vec{e_z}$

On vérifie : $\vec{a_e}(M, R/F) = \vec{a}(O/F) + \dfrac{d\vec{\Omega}(R/F)}{dt} \wedge \vec{OM} + \vec{\Omega}(R/F) \wedge (\vec{\Omega}(R/F) \wedge \vec{OM})$

$= \vec{0} + \vec{0} + \omega\vec{e_z} \wedge (\omega\vec{e_z} \wedge r\vec{e_1})$

1

$= \omega\vec{e_z} \wedge \omega r\vec{e_\varphi} = -\omega^2 r\vec{e_\rho} = -\omega^2 R\sin\alpha\, \vec{e_\rho}$

$\boxed{= \vec{a}(M/F)}$

(d) $m\vec{a}(M/R) = \vec{0} = \sum \vec{F} = \vec{P} + \vec{T} + \vec{F_{ie}} + \vec{F_{ic}}$.

1

$\vec{F_{ie}} = -m\vec{a_e}(M, R/F)$ et $\vec{F_{ic}} = -m\vec{a_c}(M, R/F)$

$= -m\vec{a}(M/F)$ $\qquad = 0$

$\Rightarrow \vec{P} + \vec{T} - m\vec{a}(M/F) = \vec{0} \Rightarrow \boxed{m\vec{a}(M/F) = \vec{P} + \vec{T}}$

3. (a) $\vec{v}(M/F) = \omega R \sin\alpha \, \vec{u_y}$. 0,5

0,5 (b) $m\vec{a}(M/F) = \sum \vec{F} = \vec{P}$ ($\vec{T} = \vec{0}$ car contact avec tige rompu).

1 $\Leftrightarrow m \begin{vmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ -mg \end{vmatrix} \Leftrightarrow \begin{vmatrix} \dot{x} = cte \\ \dot{y} = cte \\ \dot{z} = -gt + cte. \end{vmatrix}$ à $t = t_1$: $\vec{v}(M/F, t_1) = \omega R \sin\alpha \, \vec{u_y}$

1 $\Rightarrow \begin{vmatrix} \dot{x} = 0 \\ \dot{y} = \omega R \sin\alpha \\ \dot{z} = g(t_1 - t) \end{vmatrix} \Rightarrow \begin{vmatrix} x = cte \\ y = (\omega R \sin\alpha)t + cte \\ z = -\frac{g}{2}t^2 + g t_1 t + cte. \end{vmatrix}$ à $t = t_1$: $M(d, 0, 0)$

1,5 $\Rightarrow \begin{vmatrix} x = d \\ y = (\omega R \sin\alpha)t - (\omega R \sin\alpha)t_1 = \omega R \sin\alpha (t - t_1). \\ z = -\frac{g t^2}{2} + g t_1 t - \frac{g t_1^2}{2} = -\frac{g}{2}(t - t_1)^2 \end{vmatrix}$

On a $t - t_1 = \frac{y}{\omega R \sin\alpha}$ $\Rightarrow$ $\boxed{z = -\frac{g}{2} \frac{y^2}{\omega^2 R^2 \sin^2\alpha}}$ $\Rightarrow$ $\boxed{\text{parabole}}$ 0,5

1

(c) Asymptote : verticale Oz. 0,5

(d) Au sol : $z = L - OP = L - R\cos\alpha$ 0,5

$\Leftrightarrow y_{sol} = \left[-\frac{2\omega^2 R^2 \sin^2\alpha}{g} \cdot z_{sol} \right]^{\frac{1}{2}} = \boxed{\sqrt{\frac{2\omega^2 R^2 \sin^2\alpha \, (L - R\cos\alpha)}{g}}}$

1

(e) $m\vec{a}(M/F) = \vec{P} + \vec{F} = -mg\vec{z} - f\vec{v}(M/F)$.

$\Leftrightarrow m \begin{vmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ -mg \end{vmatrix} - f \begin{vmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{vmatrix}$

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$$\Leftrightarrow \begin{cases} \ddot{x} + \frac{\lambda}{m}\dot{x} = 0 \\ \ddot{y} + \frac{\lambda}{m}\dot{y} = 0 \\ \ddot{z} + \frac{\lambda}{m}\dot{z} + g = 0 \end{cases} \Leftrightarrow \begin{cases} \ddot{x} = 0 \\ \frac{\ddot{y}}{\dot{y}} = -\frac{\lambda}{m} \\ \ddot{z} + \frac{\lambda}{m}\dot{z} = -g \end{cases} \Leftrightarrow \begin{cases} \dot{x} = cte \\ \dot{y} = \dot{y}_0 e^{-\frac{\lambda}{m}t} \\ \dot{z} = \dot{z}_0 e^{-\frac{\lambda}{m}t} - \frac{mg}{\lambda} \end{cases}$$

<u>C.I.:</u>

$$\begin{cases} \dot{x}(t=t_1) = 0 \\ \dot{y}(t=t_1) = \omega R \sin\alpha \\ \dot{z}(t=t_1) = 0 \end{cases} \Rightarrow \begin{cases} cte = 0 \\ \dot{y}_0 = \omega R \sin\alpha . e^{\frac{\lambda}{m}t_1} \\ \dot{z}_0 = \frac{mg}{\lambda} e^{\frac{\lambda}{m}t_1} \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x} = 0 \\ \dot{y} = \omega R \sin\alpha\, e^{-\frac{\lambda}{m}(t-t_1)} \\ \dot{z} = \frac{mg}{\lambda}\left[e^{-\frac{\lambda}{m}(t-t_1)} - 1\right] \end{cases} \Rightarrow \begin{cases} x = cte \\ y = -\frac{m}{\lambda}\omega R \sin\alpha\, e^{-\frac{\lambda}{m}(t-t_1)} + cte \\ z = -\frac{m^2 g}{\lambda^2} e^{-\frac{\lambda}{m}(t-t_1)} - \frac{mg}{\lambda}t + cte \end{cases}$$

<u>C.I.:</u> à $t = t_1$: $M(d, 0, 0)$.

$$\Rightarrow \begin{cases} x = d \\ y = \frac{m}{\lambda}\omega R \sin\alpha\left[1 - e^{-\frac{\lambda}{m}(t-t_1)}\right] & \left[cte = \frac{m}{\lambda}\omega R\sin\alpha\right] \\ z = \frac{m^2 g}{\lambda^2}\left[1 - e^{-\frac{\lambda}{m}(t-t_1)}\right] - \frac{mg}{\lambda}(t-t_1) & \left[cte = \frac{m^2 g}{\lambda^2} + \frac{mg}{\lambda}t_1\right] \end{cases}$$

$$z = \frac{m^2 g}{\lambda^2}\frac{y}{\frac{m}{\lambda}\omega R\sin\alpha} - \frac{mg}{\lambda}(t-t_1) = \frac{mg}{\lambda}\frac{y}{\omega R \sin\alpha} - \frac{mg}{\lambda}(t-t_1).$$

$$t - t_1 = -\frac{m}{\lambda}\ln\left[1 - \frac{y}{\left(\frac{m}{\lambda}\omega R\sin\alpha\right)}\right]$$

$$\Rightarrow z = \frac{mg}{\lambda}\frac{y}{\omega R\sin\alpha} + \frac{m^2}{\lambda^2}g\ln\left[1 - \frac{y}{\frac{m}{\lambda}\omega R\sin\alpha}\right]$$

$$z = \frac{m^2}{\gamma^2} g \left[\frac{y}{\frac{m}{\gamma}\omega R \sin\alpha} + \ln\left[1 - \frac{y}{\frac{m}{\gamma}\omega R \sin\alpha}\right]\right]$$

$$\boxed{z = \frac{m^2}{\gamma^2} g \left[Y + \ln(1 - Y) \right]} \quad \text{où } Y = \frac{y}{\frac{m}{\gamma}\omega R \sin\alpha}$$

<u>Asymptote</u> : verticale Oz.